Conjugacy classes of reflections of maps

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This talk considers how many conjugacy classes of reflections a map can have, under various transitivity conditions. It is well known that regular maps have at most three classes of reflections. I shall show that for vertex- and for face-transitive maps there is no restriction on their number or size, whereas edge-transitive maps can have at most four classes of reflections. Examples are constructed, using topology, covering spaces and group theory, to show that various distributions of reflections can be achieved.

Motivation from Riemann surfaces and algebraic curves

A compact Riemann surface $\mathcal{S}$ can be regarded as a complex algebraic curve $\mathcal{C}$. The conjugacy classes of orientation-reversing involutions of $\mathcal{S}$ correspond to the real forms of $\mathcal{C}$, and the reflections correspond to those with real points. By Belyĭ's Theorem $\mathcal{C}$ is defined over an algebraic number field if and only the complex structure on $\mathcal{S}$ it is obtained from a map ($=$ dessin d'enfant $=$ graph embedding).

There are connections with work by Natanzon, by Bujalance, Gromadzki and Izquierdo, and by Bujalance and Singerman on symmetries of Riemann surfaces, and by Melekoğlu and Singerman on patterns of reflections of regular maps.

Maps and reflections

A *map* $\mathcal{M}$ is (for this talk) an embedding of a graph $\mathcal{G}$ (finite, connected, possibly with loops and multiple edges) in a surface $\mathcal{S}$ (compact, connected, oriented, without boundary), so that the faces (connected components of $\mathcal{S} \setminus \mathcal{G}$) are homeomorphic to discs.

A *reflection* of a map $\mathcal{M}$ is an automorphism which fixes a point p and acts as a euclidean reflection on some neighbourhood of p; one can choose p to be a vertex or the midpoint of an edge or face.

Reflections of orientably regular maps

Define $cr(\mathcal{M})$ to be the number of conjugacy classes of reflections in $\mathrm{Aut}\,\mathcal{M}$ (i.e. the number of 'visibly different' reflections of $\mathcal{M}$).

If $\mathcal{M}$ is *orientably regular* (i.e. the orientation-preserving automorphism group $\mathrm{Aut}^{+}\mathcal{M}$ is transitive on directed edges) then $cr(\mathcal{M}) \leq 3$, and all values within this bound can be achieved: for instance, the tetrahedron and the cube have $cr(\mathcal{M}) = 1$ and 2, while the torus maps $\{4,4\}_{b,0}$ and $\{4,4\}_{b,c}$ ($c \neq b, 0$) have $cr(\mathcal{M}) = 3$ and 0.

The aim of this talk is to consider what can be said if one relaxes the requirement of orientable regularity.

Orientably regular maps with $cr(\mathcal{M}) = 1$ and 2

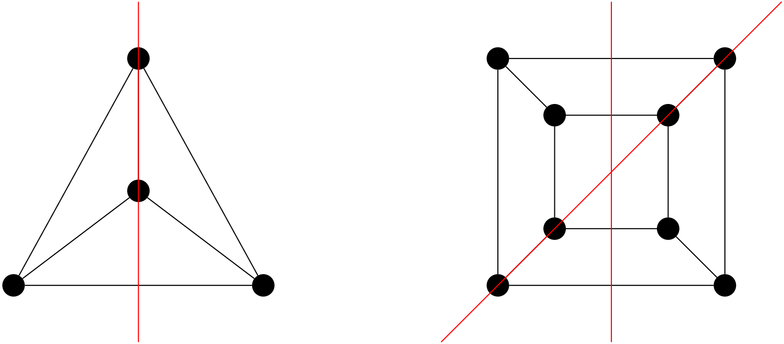


Figure : The tetrahedron and the cube

Axes of symmetry, in red, represent the conjugacy classes of reflections.

Orientably regular maps with $cr(\mathcal{M}) = 3$ and 0

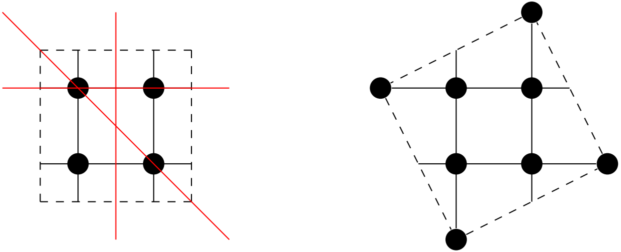


Figure : The torus maps $\{4,4\}_{2,0}$ and $\{4,4\}_{2,1}$

Identify opposite sides of each outer square to form a torus.

The torus map $\{4,4\}_{b,c}$ is the square tessellation of $\mathbb{C}/(b+ci)\mathbb{Z}[i]$.

In the case of vertex-transitive maps, there are no group-theoretic or arithmetic restrictions on the number of conjugacy classes of reflections, or on their sizes:

Theorem

Let G be a finite group with a subgroup G^+ of index 2, and let $K_1, \dots, K_k$ be distinct conjugacy classes of involutions in $G \setminus G^+$ for some $k \geq 1$. Then there is a vertex-transitive map $\mathcal{M}$, on a compact orientable surface without boundary, such that $\mathrm{Aut}\,\mathcal{M} \cong G$ and the reflections of $\mathcal{M}$ correspond to the elements of the conjugacy classes K_i.

Corollary

If $c_1, \dots, c_k$ are positive integers for some $k \geq 1$, there is a vertex-transitive map $\mathcal{M}$, on a compact orientable surface without boundary, such that the reflections of $\mathcal{M}$ form k conjugacy classes of sizes $c_1, \dots, c_k$.

Outline proof of the Theorem and Corollary

Choose $g_i \in K_i$ ($i = 1, \ldots, k$), and if necessary $g_{k+1}, \ldots, g_l \in G^+$ so that $\langle g_1, \ldots, g_l \rangle = G$. Let C be the Cayley graph for G, with single undirected edges between g and gg_i ($i \leq k$), but two reverse directed edges for any involutions g_i ($i > k$). Then G acts on C by $h : g \mapsto h^{-1}g$, with each $h = gg_ig^{-1} \in K_i$ reversing edges between g and gg_i ($g \in G, i \leq k$); no other $h \in G$ has fixed points in C.

Extend this action of G to the boundary $\mathcal{S}$ of a tubular neighbourhood of C (spheres around vertices, tubes around edges).

Draw a 1-vertex map on a fundamental region F for G on S (a sphere minus discs), so that $|G|$ copies of it form a vertex-transitive map $\mathcal{M}$ on $\mathcal{S}$, with each $h \in K_i$ acting as a reflection.

For the Corollary, given $c_1, \ldots, c_k$, take $G = D_{n_1} \times \cdots \times D_{n_k}$ with $n_i = c_i$ or $2c_i$ as c_i is odd or even, and take G^+ to consist of those elements with an even number of reflections as coordinates.

Using map duality, one can replace the condition of vertex-transitivity with face-transitivity. The situation is completely different for edge-transitive maps:

Theorem

If $\mathcal{M}$ is an edge-transitive map then $cr(\mathcal{M}) \leq 4$.

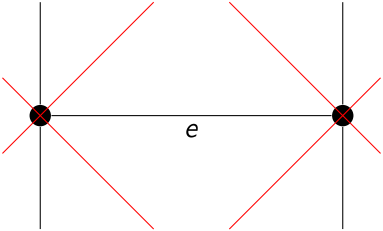


Figure : Four reflections at a typical edge e

The bound $cr(\mathcal{M}) \leq 4$ applies to all (connected) edge-transitive maps, possibly non-compact, non-orientable, or with boundary.

Examples show that $cr(\mathcal{M})$ can take any value $k \leq 4$; the maps with $cr(\mathcal{M}) = 4$ are all just-edge-transitive, i.e. neither vertex- nor face-transitive (equivalently, of automorphism type 3 in the Graver-Watkins taxonomy of edge-transitive maps). Conversely:

Theorem

If $\mathcal{M}$ is a just-edge-transitive map then $1 \leq cr(\mathcal{M}) \leq 4$.

All four values are attained by maps which can be chosen to be compact, without boundary, and orientable or non-orientable.

A just-edge-transitive map with $cr(\mathcal{M}) = 2$

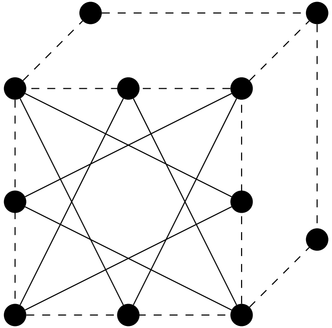
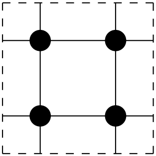


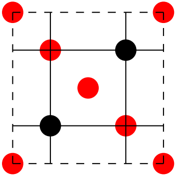
Figure : Octagonal and hexagonal faces of $\mathcal{M}$

$8 + 12 = 20$ vertices = vertices and edge-midpoints of a cube $\mathcal{C}$.
$6 \times 8 = 48$ edges = 'knight's moves'.
$6 + 8 = 14$ octagonal and hexagonal faces shown above.
$\chi = -14$, $g = 8$, $\mathrm{Aut}\,\mathcal{M} = \mathrm{Aut}\,\mathcal{C} \cong S_4 \times C_2$, $cr(\mathcal{M}) = cr(\mathcal{C}) = 2$.

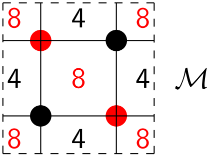
A just-edge-transitive map with $cr(\mathcal{M}) = 4$



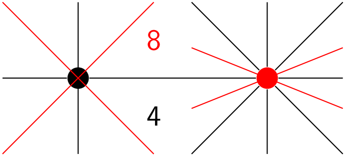
$\{4,4\}_{2,0}$



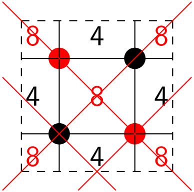
branch-points



4-gons and 8-gons



reflections (local)



reflections (global)

$\mathcal{M}$ is a double cover of the torus map $\{4,4\}_{b,0}$ ($=$ square tessellation of $\mathbb{C}/b\mathbb{Z}[i]$), for b even, branched over alternate vertices and face-centres (red dots). The genus is $1 + b^2/2$.

Types of reflections

A *flag* is a mutually incident vertex-edge-face triple.

For each $i = 0, 1, 2$, a reflection has *type i* if it transposes the i-dimensional components of some pair of flags while preserving their j-dimensional components for $j \neq i$.

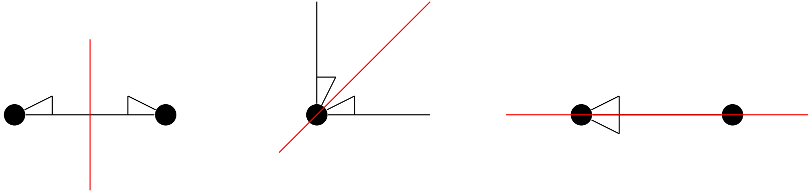


Figure : Reflections of types $0, 1, 2$

Each reflection has at least one type, and may have more than one!

Let $cr_i(\mathcal{M})$ be the number of conjugacy classes of reflections of type i of a map $\mathcal{M}$, so that

$$cr(\mathcal{M}) \leq cr_0(\mathcal{M}) + cr_1(\mathcal{M}) + cr_2(\mathcal{M}).$$

(Remember: a reflection may be of more than one type.)

Theorem

Given any integers $c_0, c_1, c_2 \geq 0$, with $c_0, c_2 \geq 1$ if c_1 is odd, there is a compact map $\mathcal{M}$ with $cr_i(\mathcal{M}) = c_i$ for $i = 0, 1$ and 2, and

$$cr(\mathcal{M}) = c_0 + c_1 + c_2.$$

General algebraic method

Maps $\mathcal{M}$ correspond to permutation representations of the group

$$\Gamma := \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle \cong V_4 * C_2,$$

acting on the set Φ of flags of $\mathcal{M}$. Each generator R_i changes the i-dimensional component of a flag while fixing the other two. By the defining relations (obviously satisfied) Γ is the free product of

$$\langle R_0, R_2 \rangle \cong V_4 := C_2 \times C_2 \quad \text{and} \quad \langle R_1 \rangle \cong C_2.$$

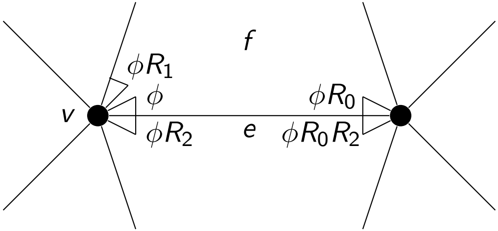


Figure : Generators R_i acting on a flag $\phi = (v, e, f)$

$\mathcal{M}$ is connected if and only if Γ is transitive on Φ, in which case $\mathcal{M}$ corresponds to a conjugacy class of *map subgroups* $M = \Gamma_\phi \leq \Gamma$, the stabilisers of flags $\phi \in \Phi$.

Then $\mathrm{Aut}\,\mathcal{M} \cong N/M$ with $N = N_\Gamma(M)$, the normaliser of M in Γ.

$\mathcal{M}$ is vertex-, edge- or face-transitive if and only if $\Gamma = ND$ where $D = \langle R_1, R_2 \rangle$, $\langle R_0, R_2 \rangle$ or $\langle R_0, R_1 \rangle$; equivalently N is transitive on the cosets of D, and vice versa.

Reflections of type i of $\mathcal{M}$ are induced by conjugates of R_i in N, equivalently fixed points of R_i in the action of Γ on cosets of N.

To construct maps $\mathcal{M}$ with given transitivity and reflection properties, find a permutation representation of Γ ($\equiv$ quotient map $\overline{\mathcal{M}}$) with D transitive and R_0, R_1, R_2 having the required fixed points. Take N to be a point-stabiliser (map subgroup for $\overline{\mathcal{M}}$), get a presentation for N (Reidemeister-Schreier algorithm), find $M \leq N$ with $N = N_\Gamma(M)$, and let $\mathcal{M}$ be the corresponding map.

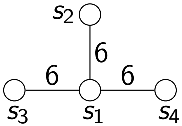
Example

To construct arbitrarily large just-edge-transitive maps $\mathcal{M}$ with $cr(\mathcal{M}) = 4$, take $N = \langle\langle R_1 \rangle\rangle^{\Gamma}$ (normal closure of R_1 in Γ), so

$$N = \langle\, S_1 = R_1,\ S_2 = R_1^{R_0},\ S_3 = R_1^{R_2},\ S_4 = R_1^{R_0 R_2} \mid S_i^2 = 1 \,\rangle$$

$$\cong C_2 * C_2 * C_2 * C_2,$$

apply an epimorphism $N \to C :=$ Coxeter group with diagram



and use a recent result of Caprace and Minasyan to show that C is conjugacy separable, so there are arbitrarily large finite images G of C (and hence of N) with the images of the reflections S_i in distinct conjugacy classes. The maps $\mathcal{M}$ corresponding to the kernels M of these epimorphisms $N \to G$ are as required.